K-THEORY AND APPLICATIONS || TOPOLOGY LEARNING SEMINAR

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1. Introduction and motivation

1.1. **Introduction.** Algebraic K-theory is the study of a family of functors

$$K_n\colon \textbf{Ring} \to \textbf{Ab}$$

where **Ring** denotes the category of rings and **Ab** denotes the category of abelian groups. For our purposes, we are only interested in this functor for values $n \in \{0, 1, 2\}$. More generally, we have also have a family of functors $K_n\colon \mathbb{C} \to \textbf{Ab}$ where $\mathbb{C}$ is some other category.

There functors will also have a "reduced" version which will be denoted by $\widetilde{K}_n$ and will often be more useful practically.

1.2. **Motivation, or applications to topology.** In this seminar we are particular interested in K-theory because of its applications in manifold topology. We will now briefly outline some of these.

1.2.1. *Wall's finiteness obstruction.* We say a manifold X is *finitely dominated* if there exists K a finite CW complex with maps $r\colon K \to X$, $i\colon X \to K$ such that $r \circ i \simeq \mathrm{Id}_X$.

Then for any finitely dominated X there exists an element denoted $[X] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ such that X has the homotopy type of a finite CW complex if and only $[X] = 0$. Furthermore, for all $\alpha \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ there exists a finitely dominated manifold X such that $[X] = \alpha$.

1.2.2. *Siebenmann's end obstruction.* Let W^n be an open manifold. We define a *tame end*, denoted $\mathcal{E}$ to be a sequence $\{W \supset U_1 \supset U_2 \dots\}$ with $\pi_1(U_1) \cong \pi_1(U_2) \cong \dots$ and each U_i connected, open and finitely dominated with $\cap_i U_i = \emptyset$. Furthermore, we say that $\mathcal{E}$ is *collared* if has a neighbourhood homeomorphic to $M^{n-1} \times [0, \infty)$ for some compact M^{n-1}.

Then for $n \geq 6$ a tame end $\mathcal{E}$ is collared if and only if $[\mathcal{E}] = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi_1(W)])$ where $[\mathcal{E}] := \lim_i [U_i]$.

1.2.3. *Whitehead torsion.* Let X and Y be finite CW complexes with $f\colon X \xrightarrow{\simeq} Y$. We say that f is *simple* if $f \simeq [X \xrightarrow{\sigma_1} X_1 \xrightarrow{\sigma_2} X_2 \xrightarrow{\sigma_3} \dots \xrightarrow{\sigma_k} X_k = Y]$ where each σ_i is a homotopy equivalence corresponding to an elementary expansion or contraction. An elementary expansion is the canonical homotopy equivalence between a complex K and K', the complex formed by adding a cancelling pair of cells to K. An elementary contraction is simply the reverse of an elementary expansion.

We can associate to any homotopy equivalence $f\colon X \xrightarrow{\simeq} Y$ a quantity named the *Whitehead torsion* $\tau(f) \in \mathrm{Wh}(\pi_1(Y))$ where $\mathrm{Wh}(\pi_1(Y))$ is defined as a quotient of $K_1(\mathbb{Z}[\pi_1(Y)])$. Then f is a simple homotopy equivalence if and only $\tau(f) = 0$.

1.2.4. *The s-cobordism theorem.* Let $(W^n; M, N)$ be an h-cobordism. That is, $\partial W = M \sqcup -N$ and both inclusions $i\colon M \overset{\simeq}{\hookrightarrow} W$ and $N \overset{\simeq}{\hookrightarrow} W$ are homotopy equivalences. Then for $n \geq 5$ we have that $W \cong M \times I$ if and only if $\tau(i) = 0$. Moreover, for any group G and for all $x \in \mathrm{Wh}(G)$ there exists an h-cobordism (W^n, M, N) with $\pi_1(X) \cong G$ such that $\tau(i\colon M \hookrightarrow W) = x$.

1.2.5. *Pseudo-isotopy.* Let M^n be a smooth manifold with $n \geq 4$ and let $f_0, f_1\colon M \to M$ be two self-diffeomorphisms. A *pseudo-isotopy* between f and g is a diffeomorphism $F\colon M \times I \to M \times I$ such that $F_{M \times \{i\}} = f_i \times \mathrm{Id}_{\{} i\}$. We can associate to any such pseudo-isotopy an element $\Sigma(F) \in \mathrm{Wh}_2(\pi_1(M))$ where $\mathrm{Wh}_2(\pi_1(M))$ is defined as a quotient of $K_2(\mathbb{Z}[\pi_1(M)])$. Then F is isotopic to an isotopy between f_0 and f_1 only if $\Sigma(F) = 0$. Note that here the reverse implication does not hold in general, and that there is a secondary obstruction that will not be mentioned here, although it is also K-theoretic in nature.

1.2.6. *Topological K-theory.* Coming soon...

2. PROJECTIVE MODULES AND K_0

2.1. Definition of K_0.

In all that follows let Λ be a ring (with unit) and all of our modules will be left Λ-modules unless stated otherwise.

Definition 2.1. We say that a module M is *projective* if for any modules A and B as shown below

$$\begin{tikzcd} & M \arrow[d] \arrow[dl, dotted] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

there exists the dotted map above $M \to A$ such that the diagram commutes.

Equivalently, M is a projective module if there exists a module N such that $M \oplus N$ is a free module.

Definition 2.2. We now define the *projective module group* $K_0(\Lambda)$, which we define formally as the abelian group having generators $[P]$ for all finitely generated projective modules P, and relations given by $[P \oplus Q] = [P] + [Q]$. Equivalently $K_0(\Lambda)$ can be defined as the formal differences of modules (with the same group operation) in the following manner.

$$K_0(\Lambda) = \left\{ \frac{[P] - [Q]}{[P] - [Q] \sim [P'] - [Q']} \right\}$$

where P', Q' are any modules satisfying $P \oplus Q' \oplus F \cong P' \oplus Q \oplus F$ for some free module F.

If Λ is commutative then we can impose more structure on $K_0(\Lambda)$. In this case, define a product in the projective module group as $[P] \cdot [Q] := [P \otimes_\Lambda Q]$. This turns $K_0(\Lambda)$ into a commutative ring.

Example 2.3. Since all projective $\mathbb{Z}$-modules (projective abelian groups) are free, we have the following:

$$\begin{aligned} K_0(\Lambda) &\xrightarrow{\cong} \mathbb{Z} \\ [\mathbb{Z}^n] &\mapsto n. \end{aligned}$$

In fact, if Λ is a field, division ring, principal ideal domain, or a local ring then we similarly have that all projective modules are free. Hence, in all of these cases we also have $K_0(\Lambda) \cong \mathbb{Z}$.

2.2. **Functoriality.** Let $f \colon \Lambda \to \Lambda'$ be a ring homomorphism and let M be a Λ-module. We can then construct a Λ'-module $f_*(M) := \Lambda' \otimes_\Lambda M$ by using f to view Λ' as a Λ-module. This construction has the nice property that if M is free then $f_*(M)$ is also. Similarly, if M is projective then $f_*(M)$ is also projective. This second fact means that f_* induces a map

$$f_* \colon K_0(\Lambda) \to K_0(\Lambda').$$

It is not hard to see that this map sends the trivial module to the trivial module and respects compositions. Hence, we see that K_0 is actually a functor **Ring** → **Ab** as claimed in the introduction.

Example 2.4. For any ring Λ we have a natural map $i \colon \mathbb{Z} \to \Lambda$ given by sending $1 \in \mathbb{Z}$ to the unit in Λ. This induces a map $i_* \colon K_0(\mathbb{Z}) \to K_0(\Lambda)$ given by sending $n \mapsto [\Lambda^n]$

Definition 2.5. Suppose that we have a map $j \colon \Lambda \to F$ where F is a field or division ring. Then we have the following sequence of maps

$$\begin{tikzcd} \mathbb{Z} \cong K_0(\mathbb{Z}) \arrow[r, "i_*"] \arrow[rr, bend right, "\mathrm{Id}_{\mathbb{Z}}"'] & K_0(\Lambda) \arrow[r, "j_*"] & K_0(F) \cong \mathbb{Z}. \end{tikzcd}$$

Therefore, we have the following identification

$$K_0(\Lambda) \cong (\ker j_*) \oplus (\operatorname{im} i_*).$$

We then define the *reduced projective module group* $\widetilde{K}_0(\Lambda) := \ker j_*$. Such a j exists if Λ is commutative, so we can certainly make this definition in those cases.

Remark 2.6. Note that in all cases we could define $\widetilde{K}_0(\Lambda)$ to be the quotient $K_0(\Lambda)/\operatorname{im}(i_*)$, but we would not necessarily have a splitting $K_0(\Lambda) = \widetilde{K}_0(\Lambda) \oplus \mathbb{Z}$.

Exercise 2.7. Show that for $\Lambda = \Lambda_1 \times \cdots \times \Lambda_k$,

$$K_0(\Lambda) \cong K_0(\Lambda_1) \times \cdots \times K_0(\Lambda_k).$$

2.3. Dedekind domains.

Definition 2.8. A ring Λ is a *Dedekind domain* if it is commutative, has no zero-divisors and has the property that for all ideals $\mathfrak{a} \subset \mathfrak{b} \lhd \Lambda$ then there exists an ideal $\mathfrak{c} \lhd \Lambda$ such that $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$.

Remark 2.9. Note that the ideal $\mathfrak{c}$ is uniquely determined (unless $\mathfrak{a} = \mathfrak{b} = 0$). Assume we have two ideals $\mathfrak{c}$ and $\mathfrak{c}'$ with $\mathfrak{a} = \mathfrak{b}\mathfrak{c} = \mathfrak{b}\mathfrak{c}'$. Let x be such that $(x) \subset \mathfrak{c}$, then there exists an ideal $\mathfrak{d}$ such that $(x) = \mathfrak{c}\mathfrak{d}$. Hence, $\mathfrak{b}\mathfrak{d}\mathfrak{c} = \mathfrak{b}\mathfrak{d}\mathfrak{c}'$, and using that Λ is commutative and has no zero-divisors we cancel the factor of (x) on both sides to conclude $\mathfrak{c} = \mathfrak{c}'$.

We now give an equivalence relation on ideals in Λ and consider the set of ideals under this relation.

Definition 2.10. Let Λ be a Dedekind domain. We say ideals $\mathfrak{a}, \mathfrak{b} \lhd \Lambda$ are equivalent, written $\mathfrak{a} \sim \mathfrak{b}$, if there exists elements $x, y \in \Lambda$ such that $x\mathfrak{a} = y\mathfrak{b}$. We call an equivalence class of ideals, denoted $\{\mathfrak{a}\}$, the *ideal class.* We then define the *ideal class group* $C(\Lambda)$ to be the set of ideal classes.

We now show that the ideal class group forms a group under multiplication. First, a lemma.

Lemma 2.11. *Let $\mathfrak{a}$ be a principal ideal. Then an ideal $\mathfrak{b}$ is a principal ideal if and only if $\mathfrak{a} \sim \mathfrak{b}$.*

Proof. First, assume $\mathfrak{b}$ is principal. This means that there exist elements $x, y \in \Lambda$ such that $\mathfrak{a} = x\Lambda, \mathfrak{b} = y\Lambda$. But then we have $y\mathfrak{a} = xy\Lambda = x\mathfrak{b}$, and so $\mathfrak{a} \sim \mathfrak{b}$. Conversely, assume that $\mathfrak{a} \sim \mathfrak{b}$. So there exists elements $x, y \in \Lambda$ such that $x\mathfrak{a} = y\mathfrak{b}$. Now we know that $\mathfrak{a}$ is a principal ideal, which implies that $x\mathfrak{a}$ is also a principal ideal. It follows that $y\mathfrak{b}$ is a principal ideal and hence $\mathfrak{b}$ is also, completing the proof. □

Proposition 2.12. *Let Λ be a Dedekind domain. The ideal class group $C(\Lambda)$ forms a group under multiplication.*

Proof. We need to show that the multiplication operation is well-defined and has a unit and inverses.

To show well-definedness, notice that $\{\mathfrak{a}\mathfrak{b}\} = \{(x\mathfrak{a})\mathfrak{b}\}$ since $(x\mathfrak{a})\mathfrak{b} = x(\mathfrak{a}\mathfrak{b})$ and hence if $x\mathfrak{a} = x'\mathfrak{a}'$ then $\{\mathfrak{a}\mathfrak{b}\} = \{(x\mathfrak{a})\mathfrak{b}\} = \{(x'\mathfrak{a})'\mathfrak{b}\} = \{\mathfrak{a}'\mathfrak{b}\}$. Using commutativity finishes the proof of well-definedness.

The unit for the multiplication is given by (x) for any element $x \in \Lambda$. The fact that this element is well-defined and acts as a unit under multiplication follows directly from Lemma 2.11.

Inverses exist from the properties of Dedekind domains. Let $x \in \mathfrak{a} \lhd \Lambda$. Then there exists an ideal $\mathfrak{b} \lhd \Lambda$ such that $\mathfrak{a}\mathfrak{b} = (x)$, and such a $\mathfrak{b}$ is unique (see Remark 2.9). □

We now relate these ideas for ideals in a Dedekind domain Λ to projective Λ-modules with a lemma and proposition.

Lemma 2.13. *Let Λ be a Dedekind domain and $\mathfrak{a}, \mathfrak{b} \lhd \Lambda$ be ideals. Then $\mathfrak{a} \sim \mathfrak{b}$ if and only if $\mathfrak{a} \cong \mathfrak{b}$ as Λ-modules.*

Proof. First assume $\mathfrak{a} \sim \mathfrak{b}$, and so there exist $x, y \in \Lambda$ such that $x\mathfrak{a} = y\mathfrak{b}$. We then have the Λ-module isomorphism $\mathfrak{a} \cong x\mathfrak{a}$ induced by the map $z \mapsto xz$. We conclude that $\mathfrak{a} \cong \mathfrak{b}$ as Λ-modules.

Now assume that we have a Λ-module isomorphism $\varphi\colon \mathfrak{a} \to \mathfrak{b}$. Now let $x \in \mathfrak{a}$. We have that $x\mathfrak{b} = x\varphi(\mathfrak{a}) = \varphi(x\mathfrak{a}) = \varphi(x)\mathfrak{a}$, and hence that $\mathfrak{a} \sim \mathfrak{b}$. Note that here in the final equality we have used the fact that $\varphi(xz) = \varphi(x)z$ for all $z \in \mathfrak{a}$. ☐

Proposition 2.14. *Let Λ be a Dedekind domain. Then we have the following:*

1. *Let $\mathfrak{a} \lhd \Lambda$ be an ideal. Then $\mathfrak{a}$ is a finitely generated projective Λ-module.*
2. *Let P be a finitely generated projective Λ-module. Then there exist $\mathfrak{a}_1, \dots, \mathfrak{a}_k \lhd \Lambda$ such that $P \cong \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k$.*

Proof. We start with (1). Let $(x), \mathfrak{b} \lhd \Lambda$ be such that $(x) = \mathfrak{a}\mathfrak{b}$. Clearly $x \in \mathfrak{a}\mathfrak{b}$ and so we can write x as $x = y_1z_1 + \cdots + y_kz_k$ with $y_i \in \mathfrak{a}$ and $z_i \in \mathfrak{b}$. Now define a map $f\colon \mathfrak{a} \to \Lambda^k$ which sends

$$y \mapsto \left(\frac{y_1z_1}{x}, \dots, \frac{y_kz_k}{x}\right).$$

Note that this map is defined since $y_iz_i \in (x)$ and well-defined since Λ has no zero-divisors. We also have a map $g\colon \Lambda^k$ which sends $(x_1, \dots, x_k) \mapsto y_1x_1 + \cdots y_kx_k$, and it is easy to see that $g \circ f = \mathrm{Id}_{\mathfrak{a}}$. If we define $Q := \ker(g)$, this provides a splitting $\mathfrak{a} \oplus Q \cong \Lambda^k$, and hence $\mathfrak{a}$ is a finitely generated projective Λ-module.

Now for (2). Assume P a finitely generated projective Λ-module. Then there exists a Q such that $P \oplus Q \cong \Lambda^k$ and hence we have an embedding $i\colon P \hookrightarrow \Lambda^k$. We now proceed by induction on k. If $k = 1$, then P embeds in Λ and hence P is an ideal Λ, which completes the base case. Now consider the following of sequence of maps:

$$\begin{tikzcd}
P \arrow[r, hook, "i"] \arrow[rr, bend right, "j"'] & \Lambda^k = \Lambda^{k-1} \oplus \Lambda \arrow[r, "\pi_2"] & \Lambda
\end{tikzcd}$$

where π_2 denotes the projection map onto the second factor and $j\colon P \to \Lambda$ is defined as to make the diagram commute. Now $\mathrm{im}(j) \lhd \Lambda$ is an ideal and hence a projective Λ-module by (1). Since P is also a projective Λ-module, we have the diagram:

$$\begin{tikzcd}
& \mathrm{im}(j) \arrow[d, equal] \arrow[dl, dotted] & \\
P \arrow[r] & \mathrm{im}(j) \arrow[r] & 0
\end{tikzcd}$$

and hence $P \cong \ker(j) \oplus \mathrm{im}(j)$. This means we have an embedding $\ker(j) \hookrightarrow \Lambda^{k-1}$ and our induction hypothesis we can write $\ker(j) \cong \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_{k-1}$ for some ideals $\mathfrak{a}_l$. Setting $\mathfrak{a}_k := \mathrm{im}(j)$ gives the required result. ☐

Theorem 2.15 (Steinitz)**.** *Let Λ be a Dedekind domain and let $\mathfrak{a}_1, \dots, \mathfrak{a}_r, \mathfrak{b}_1, \dots, \mathfrak{b}_s \lhd \Lambda$ be ideals. Then $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r \cong \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_s$ if and only if $r = s$ and $\{\mathfrak{a}_1 \cdots \mathfrak{a}_r\} = \{\mathfrak{b}_1 \cdots \mathfrak{b}_s\} \in C(\Lambda)$.*

This result gives an interesting way of viewing the ideal class group in terms of K-theory.

Corollary 2.16. *Let Λ be a Dedekind domain and let $K_0(\Lambda) \supset \langle[\Lambda]\rangle \cong \mathbb{Z}$ be the infinite-cyclic group generated by trivial module. Then $C(\Lambda) \cong K_0(\Lambda)/\langle[\Lambda]\rangle$.*

Proof. Let $f\colon C(\Lambda) \to K_0(\Lambda)/\langle[\Lambda]\rangle$ be the map sending $\{\mathfrak{a}\} \mapsto [\mathfrak{a}] + \mathbb{Z}$. We claim this is well-defined and an isomorphism. It is clearly well-defined, as $\mathfrak{a} \sim \mathfrak{b}$ implies that $\mathfrak{a} \cong \mathfrak{b}$ as modules by Lemma 2.13.

Now we show f is a homomorphism. This is straightforward since we have $f(\{\mathfrak{a}\mathfrak{b}\}) = [\mathfrak{a}\mathfrak{b}] + \mathbb{Z} = [\mathfrak{a}] + [\mathfrak{b}] + \mathbb{Z} = f(\{\mathfrak{a}\}) + f(\{\mathfrak{b}\})$ where for the second equality we have used Theorem 2.15.

For surjectivity, note by Proposition 2.14 we can write any projective module P as $P \cong \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k \cong \Lambda^{k-1} \oplus (\mathfrak{a}_1 \cdots \mathfrak{a}_k)$, which is clearly contained in $f(\{\mathfrak{a}_1 \cdots \mathfrak{a}_k\})$.

Finally, we show f is injective. Let $\mathfrak{a}$ and $\mathfrak{b}$ be such that $[\mathfrak{a}] + \mathbb{Z} \cong [\mathfrak{b}] + \mathbb{Z}$. This implies $\mathfrak{a} \oplus \Lambda^k \cong \mathfrak{b} \oplus \Lambda^l$ for some k and l, but by Theorem 2.15 we have that $k = l$ and that $\{\mathfrak{a}\} = \{\mathfrak{a}\Lambda \cdots \Lambda\} = \{\mathfrak{b}\Lambda \cdots \Lambda\} = \{\mathfrak{b}\}$. This completes the proof. □

Definition 2.17. Let Λ be a Dedekind domain. We can now define a *rank* map rank: $K_0(\Lambda) \to \mathbb{Z}$ which sends $[\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k] \mapsto k$ which is well-defined by Theorem 2.15. This gives an alternate way of defining reduced K_0 in a Dedekind domain. Define

$$\widetilde{K}_0(\Lambda) := \ker(\mathrm{rank}) \cong \; K_0(\Lambda)/\langle[\Lambda]\rangle \cong C(\Lambda).$$

We now conclude this section with a (partial) proof of the main theorem.

Proof of Theorem 2.15. For simplicity, we only show the forwards implication i.e. we show that $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r \cong \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_s$ implies $r = s$ and $\{\mathfrak{a}_1 \cdots \mathfrak{a}_r\} = \{\mathfrak{b}_1 \cdots \mathfrak{b}_s\}$.

Let $\varphi\colon \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r \to \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_s$ be an isomorphism of modules. We want to show that we can write this isomorphism in terms of a matrix. To see this, consider the following. Suppose we have two ideals $\mathfrak{a}, \mathfrak{b} \lhd \Lambda$ and $\theta\mathfrak{a} \to \mathfrak{b}$ a module homomorphism between them. Let $Q(\Lambda)$ denote the field of fractions of Λ. We claim that there exists a unique $q \in Q(\Lambda)$ such that for all $x \in \mathfrak{a}$ $\theta(x) = qx \in Q(\Lambda) \supset \Lambda$. To see this, pick $x_0 \in \mathfrak{a}$ with $x_0\theta(x) = \theta(x_0 x) = \theta(x_0)x \in \Lambda$. This gives

$$\theta(x) = \frac{\theta(x_0 x)}{x_0} = \frac{\theta(x_0)}{x_0}x \in Q(\Lambda).$$

Now set $q := \theta(x_0)/x_0$.

The above means that we can write our isomorphism φ as a matrix. In other words, there exists an $(r \times s)$-dimensional matrix $Q \in Q(\Lambda)^{r\times s}$ that $\varphi(\mathbf{x}) = Q\mathbf{x} \in Q(\Lambda)^s$. Since φ is an isomorphism, this matrix is invertible and hence we have a matrix $Q^{-1} \in Q(\Lambda)^{s\times r}$ which represents $\varphi^{-1}\colon \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_s \to \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$. Naturally, this gives us that $r = s$.

Now we claim that $\mathfrak{b}_1 \cdots \mathfrak{b}_r = (\det Q)\mathfrak{a}_1 \cdots \mathfrak{a}_r$, which would imply that $\{\mathfrak{a}_1 \cdots \mathfrak{a}_r\} = \{\mathfrak{b}_1 \cdots \mathfrak{b}_r\}$ and hence complete the proof. Let $x_i \in \mathfrak{a}_i$. We then have

$$(\det Q)x_1 \cdots x_r = \det\left(Q\begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_r \end{bmatrix}\right) = \det\left(\varphi\begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \varphi\begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_r \end{pmatrix}\right) \in \mathfrak{b}_1 \cdots \mathfrak{b}_r.$$

This implies that $(\det Q)\mathfrak{a}_1 \cdots \mathfrak{a}_r \subset \mathfrak{b}_1 \cdots \mathfrak{b}_r$. The exact same argument with Q^{-1} rather than Q implies the other inclusion $\mathfrak{b}_1 \cdots \mathfrak{b}_r \subset (\det Q)\mathfrak{a}_1 \cdots \mathfrak{a}_r$, and hence the claim is proved, completing the proof. □

3. Constructing projective modules

In this section we will describe a method for producing new projective modules. First, some preliminaries. Suppose we have the following pullback square of rings such that at least one of j_1 or j_2 is surjective:

(3.1)
$$\begin{tikzcd}
\Lambda \arrow[r, "i_1"] \arrow[d, "i_2"] & \Lambda_1 \arrow[d, "j_1"] \\
\Lambda_2 \arrow[r, "j_2"] & \Lambda'
\end{tikzcd}$$

This being a pullback square implies that

$$\Lambda \cong \Lambda_1 \times_{\Lambda'} \Lambda_2 := \{(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2 \mid j_1(\lambda_1) = j_2(\lambda_2)\}.$$

Recall from Section 2.2 we have that for any ring homomorphism $f \colon \Lambda \to \Lambda'$ and any left Λ-module we have another left Λ-module $f_\sharp(M) := \Lambda_1 \otimes_\Lambda M$. We also have a map $f_* \colon M \to f_\sharp M$ given by sending $m \mapsto 1 \otimes m$. The map f_* is clearly Λ-linear as

$$\lambda m \mapsto 1 \otimes \lambda m = f(\lambda) \otimes m = \lambda \cdot (1 \otimes m).$$

Definition 3.1. Suppose we are in the situation given in Equation (3.1). Let P_k be a projective Λ_k-module for $k \in 0, 1$ and let $h : (j_1)_\sharp P_1 \to^{\cong} (j_2)_\sharp P_2$ be a Λ'-module isomorphism. We define a new module M as

$$M(P_1, P_2, h) = \{(p_1, p_2) \in P_1 \times P_2 \mid h(j_1)_*(p_1) = (j_2)_*(p_2)\}.$$

This means that we have the analogous pullback square

$$\begin{tikzcd}
M(P_1, P_2, h) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & P_1 \arrow[d, "h(j_1)_*"] \\
P_2 \arrow[r, "(j_2)_*"] & (j_2)_\sharp P_2
\end{tikzcd}$$

where π_k denote the obvious projection maps onto the respective factors. This pullback square satisfies the similar property that either $(j_2)_*$ or $h(j_1)_*$ is surjective. Note that there is a natural Λ-module structure on M given by $\lambda \cdot (p_1, p_2) = (i_1(\lambda) \cdot p_1, i_2(\lambda) \cdot p_2)$.

Remark 3.2. There exists a natural isomorphism from $M(P_1, P_2, h)$ to $M(P_2, P_1, h^{-1})$ and so the apparent asymmetry between P_1 and P_2 in Definition 3.1 is not important. This means that without loss of generality we can assume that either map is surjective.

We now state three theorems about this construction.

Theorem 3.3. *The constructed module $M(P_1, P_2, h)$ is projective as a Λ-module. Moreover, if P_1 and P_2 are finitely generated, then M is also finitely generated.*

Theorem 3.4. *Every projective Λ-module is isomorphic to $M(P_1, P_2, h)$ for some P_1, P_2 and h.*

Theorem 3.5. *If $M = M(P_1, P_2, h)$, then P_1 and P_2 are naturally isomorphic to $(i_1)_\sharp M$ and $(i_2)_\sharp M$, respectively.*

3.1. **Proof of Theorem 3.3.** We will slowly work our way up to the proof by proving weaker versions. To start, let us suppose that P_1 and P_2 are free modules. We aim to show that in this case M is projective. Let $\{x_\alpha\}$ and $\{y_\beta\}$ be bases for P_1 and P_2, respectively. It follows that $\{(j_1)_* x_\alpha\}$ and $\{(j_2)_* y_\beta\}$ are bases for $(j_1)_\sharp P_1$ and $(j_2)_\sharp P_2$, respectively. We then write $A = (a_{\alpha\beta})$ for the matrix representing the isomorphism $h\colon (j_1)_\sharp P_1 \to (j_2)\sharp P_2$. Note that A may be infinite but only has finitely many non-zero entries in each row. In terms of the bases, this means that

$$h\colon (j_1)_* x_\alpha \mapsto \Sigma a_{\alpha\beta}(j_2)_* y_\beta.$$

We also note that since h is an isomorphism, A must be invertible and we denote the inverse in the usual way as A^{-1}.

Lemma 3.6. *If the matrix A as above is the image entry-wise under j_2 of an invertible matrix over Λ_2, then $M(P_1, P_2, h)$ is free.*

Proof. Assume $a_{\alpha\beta} = j_2(c_{\alpha\beta})$ for some $C = (c_{\alpha\beta})$ an invertible matrix over Λ_2. Define elements $y'_\alpha := \Sigma c_{\alpha\beta} y_\beta \in P_2$. We would like to write down a basis for M. First, we perform the following calculation:

$$\begin{aligned} h(j_1)_* x_\alpha &= \Sigma a_{\alpha\beta}(j_2)_* y_\beta \\ &= \Sigma j_2(c_{\alpha\beta})(j_2)_* y_\beta \\ &= (j_2)_*(\Sigma c_{\alpha\beta} y_\beta) = (j_2)_* y'_\alpha \end{aligned}$$

This calculation tells us that (x_α, y'_α) is an element of M. We then define a set of elements $z_\alpha := (x_\alpha, y'_\alpha) \in M \subset P_1 \times P_2$. We claim that $\{z_\alpha\}$ is a basis for M and hence M is free.

First assume that there is a relation between the z_α, i.e. there exist elements λ_α such that $\Sigma \lambda_\alpha z_\alpha = 0$. This implies that $\Sigma i_1(\lambda_\alpha) x_\alpha = 0$ and $\Sigma i_2(\lambda_\alpha) y'_\alpha = 0$. However, since the x_α and y'_α form bases of P_1 and P_2, this implies that $i_k \lambda_\alpha = 0$ for all α and $k \in \{0, 1\}$. Hence, $\lambda_\alpha = 0$ for all α.

To finish we show that $\{z_\alpha\}$ is a spanning set. Let $(\Sigma \lambda_\alpha x_\alpha, \Sigma \mu_\alpha y'_\alpha)$ be an element of $P_1 \times P_2$ such that $h(j_1)_*(\Sigma \lambda_\alpha x_\alpha) = (j_2)_*(\Sigma \mu_\alpha y'_\alpha)$. This implies

$$\Sigma j_1(\lambda_\alpha) h(j_1)_* x_\alpha = \Sigma j_2(\mu_\alpha)(j_2)_* y'_\alpha.$$

Linear independence then gives us that $j_1(\lambda_\alpha) = j_2(\mu_\alpha)$. Then by the pullback square there must exist $\gamma_\alpha \in \Lambda$ such that $(i_1)\gamma_\alpha = \lambda_\alpha$ and $(i_2)\gamma_\alpha$. Hence we have that

$$\Sigma \gamma_\alpha z_\alpha = (\Sigma \lambda_\alpha x_\alpha, \Sigma \mu_\alpha y'_\alpha)$$

and so $\{z_\alpha\}$ is spanning, finishing the proof. □

Lemma 3.7. *Let P_1, P_2 be free and $j_2\colon \Lambda_2 \to \Lambda'$ be surjective. Then $M(P_1, P_2, h)$ is a projective Λ-module.*

Proof. Let Q_1 be free over Λ_1 with basis $\{\mu_\beta\}$ in a formal one-to-one correspondence with $\{y_\beta\}$ a basis for P_2. Similarly, let Q_2 be a free over Λ_2 with a basis $\{\nu_\alpha\}$ corresponding to $\{x_\alpha\}$ a basis for P_1. Let

$$g\colon (j_1)_\sharp Q_1 \to (j_2)_\sharp Q_2$$

be the isomorphism of Λ-modules represented by the matrix A^{-1}, where A represents h. Observe that

$$M(P_1, P_2, h) \oplus M(Q_1, Q_2, g) \cong M(P_1 \oplus P_2, Q_1 \oplus Q_2, h \oplus g).$$

We claim that $h \oplus g$ satisfies Lemma 3.6. This would finish the proof since then $M(P_1, P_2, h)$ would be a direct summand of a free module. We finish by proving the claim. Denote by I the identity matrix (of appropriate rank) and consider the following block matrix equation:

$$\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & A^{-1} \end{bmatrix} = \begin{bmatrix} I & A \\ \mathbf{0} & -I \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ -A^{-1} & I \end{bmatrix} \begin{bmatrix} I & A \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \mathbf{0} & -I \\ I & \mathbf{0} \end{bmatrix}.$$

Each factor in the right-hand side of the equation clearly is the image under j_2 of some matrix in Λ_2 and these matrices all have the forms:

$$\begin{bmatrix} I & * \\ \mathbf{0} & -I \end{bmatrix}, \begin{bmatrix} I & \mathbf{0} \\ * & I \end{bmatrix}, \begin{bmatrix} I & * \\ \mathbf{0} & I \end{bmatrix}, \begin{bmatrix} \mathbf{0} & -I \\ I & \mathbf{0} \end{bmatrix}$$

and all of these are invertible. Hence the claim is proved. □

We now move towards the general case where P_1 and P_2 are projective and the proof of Theorem 3.3, but first we state a lemma without proof.

Lemma 3.8. *Let P_1 be projective over Λ_1 and P_2 projective over Λ_2. Then there exist projective modules Q_k over Λ_k for $k \in \{1, 2\}$ such that $P_1 \oplus Q_1$ and $P_2 \oplus Q_2$ are both free. Furthermore, there exists an isomorphism $k\colon (j_1)_\sharp Q_1 \xrightarrow{\cong} (j_2)_\sharp Q_2$.*

Proof of Theorem 3.3. We start by choosing Q_1, Q_2 as given to us by Lemma 3.8. We then have that

$$M(P_1, P_2, h) \oplus M(Q_1, Q_2, k) \cong M(P_1 \oplus P_2, Q_1 \oplus Q_2, h \oplus k).$$

and Lemma 3.7 gives us that the right-hand side is projective. Hence M is a summand of a projective module which implies M is a summand of a free module and therefore is projective, completing the proof. □

4. K_1 AND THE WHITEHEAD GROUP

4.1. **Definition of K_1.** Let Λ be a ring and let $\mathrm{GL}(n, \Lambda)$ denote the group of $n \times n$ invertible matrices with entries in Λ. We have inclusions

$$\mathrm{GL}(1, \Lambda) \subset \mathrm{GL}(2, \Lambda) \subset \mathrm{GL}(3, \Lambda) \subset \dots$$

where the inclusion map is given by sending

$$\mathrm{GL}(n, \Lambda) \ni A \mapsto \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \in \mathrm{GL}(n+1, \Lambda).$$

We then define $\mathrm{GL}(\Lambda) := \cup_n \mathrm{GL}(n, \Lambda)$ to be the limit.

Definition 4.1. We say that a matrix $E \in \mathrm{GL}(\Lambda)$ is *elementary* if has only one non-zero off-diagonal entry and all of its diagonal entries are equal to 1. For $i \neq j$ and $a \neq 0$ we write $E_{ij}^n(a)$ for the elementary matrix of dimension n whose single non-zero off-diagonal entry is equal to a. Let $E(\Lambda) \subset \mathrm{GL}(\Lambda)$ be the subgroup generated by elementary matrices.

Remark 4.2. Multiplication by an elementary matrix corresponds to performing an elementary row or column operation, and every row or column operation can be realised via multiplication by an elementary matrix. Left multiplication corresponds to row operations, and right multiplication corresponds to column operation. Hence, a matrix can be written as a product of elementary matrices if and only if it can be reduced to the identity matrix by a finite sequence of row and column operations.

Lemma 4.3. *The subgroup generated by elementary matrices* $E(\Lambda)$ *is equal to the commutator subgroup, i.e.*

$$E(\Lambda) = [\mathrm{GL}(\Lambda), \mathrm{GL}(\Lambda)].$$

Proof. Basic matrix multiplication gives the following equality for $i \neq l$ and $n \geq 3$:

$$[E^n_{ij}(a), E^n_{kl}(b)] = \begin{cases} E^n_{il}(ab) & \text{if j=k} \\ I & \text{else.} \end{cases}$$

This proves that every elementary is a commutator (of elementary matrices). For the reverse inclusion, represent the commutator $[A, B] = ABA^{-1}B^{-1}$ as the block matrix

$$\begin{bmatrix} ABA^{-1}B^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} = \begin{bmatrix} AB & \mathbf{0} \\ \mathbf{0} & A^{-1}B^{-1} \end{bmatrix} \begin{bmatrix} (BA)^{-1} & \mathbf{0} \\ \mathbf{0} & BA \end{bmatrix}$$
$$= \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & A^{-1} \end{bmatrix} \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & B^{-1} \end{bmatrix} \begin{bmatrix} (BA)^{-1} & \mathbf{0} \\ \mathbf{0} & BA \end{bmatrix}.$$

As was noted in the proof of Lemma 3.7, we can write each of the matrices in the above product in the form

$$\begin{bmatrix} I & * \\ \mathbf{0} & -I \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ * & I \end{bmatrix} \begin{bmatrix} I & * \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \mathbf{0} & -I \\ I & \mathbf{0} \end{bmatrix}.$$

It is not hard to see that we can transform all of the above matrices into the identity matrix via a sequence of row and column operations, and hence the commutator can be written as a product of elementary matrices, completing the proof. □

As an immediate consequence of this, we see that $E(\Lambda)$ is a normal subgroup. We now define K_1.

Definition 4.4. We define the *torsion group* $K_1(\Lambda) := \mathrm{GL}(\Lambda)/E(\Lambda)$. This is well-defined and abelian by Lemma 4.3.

Exercise 4.5. Show that K_1 is a covariant functor by defining the obvious map $K_1(\Lambda) \to K_1(\Lambda')$ given a ring homomorphism $\Lambda \to \Lambda'$.

We can say more about $K_1(\Lambda)$ if Λ is commutative. Naturally, assume now that Λ is commutative. The determinant map $\det\colon \mathrm{GL}(\Lambda) \to \Lambda^\times$ is now well-defined we define $\mathrm{SL}(\Lambda)$ to be the kernel of this map. We then have the exact sequence

$$\mathrm{SL}(\Lambda)/E(\Lambda) \to K_1(\Lambda) \xrightarrow{\det} \Lambda^\times$$

which splits. To see this, note that we have an inclusion $\Lambda^\times = \mathrm{GL}(1, \Lambda) \subset \mathrm{GL}(\Lambda)$ and the inclusion post-composed with det is clearly the identity map. This means we have an isomorphism

$$K_1(\Lambda) = \Lambda^\times \oplus \mathrm{S}K(\Lambda)$$

where we have defined $\mathrm{S}K_1(\Lambda) = \mathrm{SL}(\Lambda)/E(\Lambda)$.

Example 4.6. In many cases, $\mathrm{S}K(\Lambda)$ vanishes and so $K_1(\Lambda) = \Lambda^\times$. Here are some examples where that occurs.

- If Λ is a field.
- If Λ is $\mathbb{Z}$ or any Euclidean domain.
- If Λ has only finitely many ideals.
- If Λ is the ring of a integers of a finite extension of $\mathbb{Q}$.
- If Λ is $\mathbb{Z}[\mathbb{Z}/p]$ for p a prime.

However, $\mathrm{S}K_1(\Lambda)$ does not always vanish. Let $\Lambda = \mathbb{R}[X,Y]/(X^2+Y^2-1)$ where X, Y are abstract variables, then $\mathrm{S}K_1(\Lambda) \cong \mathbb{Z}/2$ generated by the matrix $\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$.

4.2. **The Whitehead group.** We now turn briefly to the Whitehead group, which was mentioned (not by name) in Section 1.2.3. Let

$$\tau\colon\ \mathrm{GL}(n,\Lambda) \to K_1(\Lambda)$$

be the obvious map given by the inclusion followed taking the quotient. We call this the *torsion* map.

Definition 4.7. We now define the *reduced torsion group*

$$\overline{K}_1(\Lambda) := K_1(\Lambda)/\{\tau(-1)\}$$

where we consider $-1 \in \mathrm{GL}(1,\Lambda)$. Furthermore, let π be a group and let $\Lambda = \mathbb{Z}[\pi]$. We then define the *Whitehead group*

$$\mathrm{Wh}(\pi) := K_1(\Lambda)/\{\tau(\pm g) \mid g \in \pi\}.$$

Remark 4.8. The same thing in different words is that we have a sequence of maps

$$\pi \to K_1(\mathbb{Z}[\pi]) \to \overline{K}_1(\mathbb{Z}[\pi])$$

and then the Whitehead group is the cokernel, i.e. $\mathrm{Wh}(\pi) = \overline{K}_1(\mathbb{Z}[\pi])/\operatorname{im}(\pi)$.

Recall from Section 1.2.3 that we say a homotopy equivalence of CW-complexes is simple if it can be written as a sequence of elementary expansions or contractions. We mentioned a result which said that a homotopy equivalence $f\colon X \to Y$ is simple if and only if the map's Whitehead torsion $\tau(f) = 0 \in \mathrm{Wh}(\pi)$, and we have now described the group that this obstruction lives in. Of course, we have not yet defined what Whitehead torsion is. However, if the Whitehead group vanishes then conceptually this is not a problem, so we will now give some examples where this is the case.

Example 4.9. The Whitehead group $\mathrm{Wh}(\pi)$ vanishes if

- π is $\mathbb{Z}$.
- π is $\mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4$.
- π is free abelian.
- π is free.

It is conjectured that $\mathrm{Wh}(\pi) = 0$ for all torsion-free π. Of course, this group does not always vanish. For example, $\mathrm{Wh}(\mathbb{Z}/5) \cong \mathbb{Z}$.

Example 4.10. Proving that $\mathrm{Wh}(\mathbb{Z}/5) \cong \mathbb{Z}$ is highly non-trivial, but we try to give a little idea here of why this might be true.

From Example 4.6 we know that $K_1(Z[\mathbb{Z}/5]) \cong (\mathbb{Z}[\mathbb{Z}/5])^{\times}$. Write t for the generator of $\mathbb{Z}/p$ in $\mathbb{Z}[\mathbb{Z}/p]$. Define an element $u := t+t^{-1}-1 \in \mathbb{Z}[\mathbb{Z}/5]$ and we claim that u generates $\mathrm{Wh}(\mathbb{Z}/5)$. First, we need to show that u is a unit, but this is easy as $(t+t^{-1}-1)(t^2+t^{-2}-1) = 1$. Next, we show that u has infinite order in $K_1(\mathbb{Z}[\mathbb{Z}/5])$. Define a homomorphism $\theta\colon (\mathbb{Z}[\mathbb{Z}/5]) \to \mathbb{C}$ by $t \mapsto \zeta = e^{2\pi i/5}$. Then it is not hard to see that $\theta(u) = 2\cos(72^{\circ}) - 1 \in \mathbb{R}$ and hence $\theta(u)$ has infinite order in $\mathbb{C}$.

Of course, this does not prove that u has infinite order in $\mathrm{Wh}(\mathbb{Z}/5)$, or indeed that it generates the whole group. Nevertheless, this gives some idea of why $\mathrm{Wh}(\mathbb{Z}/5)$ might be non-trivial.

4.3. A Mayer-Vietoris sequence for K_0 and K_1.

We now will relate the groups K_0 and K_1 using a 'Mayer-Vietoris' sequence, called that because it will closely resemble the familiar Mayer-Vietoris sequence from Algebraic Topology. First, assume we again have rings $\Lambda, \Lambda_1, \Lambda_2, \Lambda'$ and maps that satisfy the same pullback square in Equation (3.1). Without loss of generality assume that $j_2\colon \Lambda_2 \to \Lambda'$ is surjective. Consider the following sequence:

$$(\dagger)\qquad \begin{tikzcd} & \text{(I)} & \text{(II)} & \text{(III)} & \text{(IV)} & \\ & K_1(\Lambda_1) & & & K_0(\Lambda_1) & \\ K_1(\Lambda) \arrow[r, "\iota_1"] & \oplus \arrow[r, "q_1"] & K_1(\Lambda') \arrow[r, "\partial"] & K_0(\Lambda) \arrow[r, "\iota_0"] & \oplus \arrow[r, "q_0"] & K_0(\Lambda') \\ & K_1(\Lambda_2) & & & K_0(\Lambda_2) & \end{tikzcd}$$

where

$$\iota_\alpha\colon x \mapsto \big((i_1)_*(x), (i_2)_*(x)\big),$$
$$q_\alpha\colon (y, z) \mapsto (j_1)_*(y) - (j_2)_*(z)$$

and the 'boundary map' ∂ is defined in the following manner. Let $[X] \in K_1(\Lambda'))$ with X a matrix representative. Given standard bases for Λ_1^n and Λ_2^n, X represents an isomorphism $h\colon (j_1)_\sharp \Lambda_1^n \to (j_2)_\sharp \Lambda_2^n$. Now let $M = M(\Lambda_1^n, \Lambda_2^n, h)$, using the construction from Definition 3.1, and we define

$$\partial\colon [X] \mapsto [M] - [\Lambda^n].$$

Lemma 4.11. *The map $\partial\colon K_1(\Lambda') \to K_0(\Lambda)$ as defined above is a well-defined homomorphism.*

Proof. Proof We start with showing the map is well-defined. Let E' be some elementary matrix over Λ' and we may assume it has the same dimension as X without loss of generality. The aim is to show that the modules $M = M(\Lambda_1^n, \Lambda_2^n, h)$ and $M' = M(\Lambda_1^n, \Lambda_2^n, h')$ are isomorphic where h' is the isomorphism corresponding to XE'. Since j_2 is surjective, we can lift E' to a matrix E_2 over Λ_2 which must itself be an elementary matrix, and thus invertible. Hence, the matrix $\begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & E_2 \end{bmatrix}$ defines an map $\Lambda_1^n \times \Lambda_2^n$ which restricts to an isomorphism between M and M'.

Having shown that the map is well-defined, it is easy to see it is a homomorphism by considering the following matrix identity:

$$\begin{bmatrix} XY & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} Y^{-1} & \mathbf{0} \\ \mathbf{0} & Y \end{bmatrix} = \begin{bmatrix} X & \mathbf{0} \\ \mathbf{0} & Y \end{bmatrix}$$

and so the leftmost and rightmost equations are equivalent under multiplication by elementary matrices. This implies that $\partial([X][Y]) = \partial([XY]) = \partial([X]) \oplus \partial([Y])$, completing the proof. ☐

Theorem 4.12. *The 'Mayer-Vietoris' sequence † is exact.*

Proof (sketch). Exactness at positions (I) and (IV) follows from the properties of the pullback square Equation (3.1). Exactness at positions (II) and (III) is more interesting since it involves the boundary map ∂. We will only show exactness at position (III).

Assume $M \in \operatorname{im}\partial$, then $M = M(\Lambda_1^n, \Lambda_2^n, h)$ for some n and h. This implies that both $(i_1)_*(M)$ and $(i_2)_*(M)$ are stably-free and hence $\iota_0(M) = 0$.

Conversely, assume that $\iota_0(M) = 0$. This means that both $(i_1)_*(M)$ and $(i_2)_*(M)$ are stably-free and by Theorem 3.4 and Theorem 3.5 this implies that $M \cong M(\Lambda_1^n, \Lambda_2^n, h)$ for some n and h. If we represent h by a matrix X, then clearly $\partial[X] = M$ and so $M \in \operatorname{im}\partial$. ☐

Exercise 4.13. Complete the proof of Theorem 4.12 by showing exactness of † at positions (I), (II) and (IV).

We now show an application of the sequence † which allows us to compute $K_0(\mathbb{Z}[\mathbb{Z}/p])$. Let p be a prime and $\zeta = e^{2\pi i/p}$ be a primitive pth root of unity. Consider the following diagram.

(‡)

$$\begin{tikzcd} \mathbb{Z}[\mathbb{Z}/p] \arrow[r, "i_1"] \arrow[d, "i_2"] & \mathbb{Z}[\zeta] \arrow[d, "j_1"] \\ \mathbb{Z} \arrow[r, "j_2"] & \mathbb{Z}/p \end{tikzcd}$$

where the maps are described as follows. Write t for the generator of $\mathbb{Z}/p$ in $\mathbb{Z}[\mathbb{Z}/p]$. Then i_1 is the map that sends $t \mapsto \zeta$, i_2 is the map which sends $t \mapsto 1$, j_1 is the map which sends $\zeta \mapsto 1$ and reduces modulo p, and finally j_2 is the map which reduces modulo p. To use this square to get a Mayer-Vietoris sequence, we need that at least one of j_1 or j_2 is surjective and that this is a pullback square. The first condition is obviously true since both j_1 and j_2 are clearly surjective.

Lemma 4.14. *The square ‡ is a pullback.*

Proof. Consider the map $i\colon \mathbb{Z}[\mathbb{Z}/p] \to \mathbb{Z}[\zeta] \times \mathbb{Z}$ given by sending $x \to (i_1(x), i_2(x))$. Clearly this is an injective homomorphism. We need to show that the image of i is equal to the pullback $\{(a, b) \in \mathbb{Z}[\mathbb{Z}/p] \times \mathbb{Z} \mid j_1(a) = j_2(b)\}$. It is not hard to see that $j_1 \circ i_1(x) = j_2 \circ i_2(x)$ and so the image of i is contained in the pullback. For the other inclusion, let $(a, b) = (\Sigma_{k=0}^{p-1} \alpha_k \zeta^k, N)$ be an element in the pullback. Then set $x' = \Sigma_{k=0}^{p-1} \alpha_k t^0 \in \mathbb{Z}[\mathbb{Z}/p]$. Clearly $i_1(x') = a$, and we also have that $N - b \cong 0 \bmod p$. Let $K = N - b$ and then define $x = x + K(1 + t + t^2 + \cdots + t^{p-1})$. Since $1 + \zeta + \cdots + \zeta^{p-1} = 0$ we still have $i_1(x) = a$ but now also $i_2(x) = b$, and hence i surjects onto the pullback, completing the proof. ☐

Hence we can form the Mayer-Vietoris sequence for the square ‡. For reference, we write this out again below with some additional maps emphasised.

$$\begin{tikzcd}
 & K_1(\mathbb{Z}[\zeta]) \arrow[dr, bend left, "(j_1)_*"] & & & K_0(\mathbb{Z}[\zeta]) & \\
K_1(\mathbb{Z}[\mathbb{Z}/p]) \arrow[r, "\iota_1"] & \oplus \arrow[r, "q_1"] & K_1(\mathbb{Z}/p) \arrow[r, "\partial"] & K_0(\mathbb{Z}[\mathbb{Z}/p]) \arrow[ur, bend left, "(i_1)_*"] \arrow[r, "\iota_0"] & \oplus \arrow[r, "q_0"] & K_0(\mathbb{Z}/p) \\
 & K_1(\mathbb{Z}) & & & K_0(\mathbb{Z}) \arrow[ur, bend right, "(j_2)_*"'] &
\end{tikzcd}$$

Theorem 4.15. *The map $(i_1)_*\colon K_0(\mathbb{Z}[\mathbb{Z}/p]) \to K_0(\mathbb{Z}[\zeta])$ is an isomorphism.*

Proof. We know from Example 2.3 that $K_0(\mathbb{Z}) \cong K_0(\mathbb{Z}) \cong \mathbb{Z}$ generated by $[\mathbb{Z}]$ and $[\mathbb{Z}/p]$, respectively. This means that the map $(j_2)_*\colon K_0(\mathbb{Z}) \to K_0(\mathbb{Z}/p)$ is an isomorphism. Hence, we can conclude via exactness that $\operatorname{im}(\iota_0) = K_0(\mathbb{Z}[\zeta])$ and so to prove the theorem we have to show that $\ker(\iota_0) = 0$. Via exactness and the first isomorphism theorem for groups, this is equivalent to showing that q_1 is surjective. We will show that $(j_1)_*$ is surjective, which suffices.

Recall from Example 4.6 that we have that $K_1(\mathbb{Z}[\zeta]) \cong (\mathbb{Z}[\zeta])^\times$ and $K_1(\mathbb{Z}/p) \cong (\mathbb{Z}/p)^\times$. Hence, we need to find a unit in $\mathbb{Z}[\zeta]$ which maps to any given unit in $\mathbb{Z}/p$. Let $k \in \{0, 1, \dots, p-1\}$. Then define an element $u_k := \frac{\zeta^k - 1}{\zeta - 1} = 1 + \zeta + \dots + \zeta^{k-1}$, let $l = k^{-1} \in (\mathbb{Z}/p)^\times$ and set $\eta = \zeta^k$. The inverse for u_k is given as $v_k = \frac{\eta^l - 1}{\eta - 1} \in \mathbb{Z}[\mathbb{Z}/p]$, since

$$u_k \cdot v_k = \frac{\zeta^k - 1}{\zeta - 1} \cdot \frac{\eta^l - 1}{\eta - 1} = \frac{\zeta^k - 1}{\zeta - 1} \cdot \frac{\zeta^{kl} - 1}{\zeta^k - 1} = \frac{\zeta^k - 1}{\zeta - 1} \cdot \frac{\zeta - 1}{\zeta^k - 1} = 1.$$

Hence u_k is a unit. Then note that $(j_1)_*(u_k) = \overline{k} \in (\mathbb{Z}/p)^\times$ and hence $(j_1)_*$ is surjective, completing the proof. ☐

The upshot of this theorem is that we can understand $K_0(\mathbb{Z}[\zeta])$ fairly well since $\mathbb{Z}[\zeta]$ is a Dedekind domain. From Corollary 2.16 we have $K_0(\mathbb{Z}[\zeta]) \cong \mathbb{Z} \oplus C(\mathbb{Z}[\zeta])$ where $C(\mathbb{Z}[\zeta])$ is the ideal class group. This was studied extensively by number theorists in the 19th century as it had connections to Fermat's last theorem. We have the following table for $p \leq 50$. Omitted primes have trivial $C(\mathbb{Z}[\zeta])$.

p	$C(\mathbb{Z}[\zeta])$
23	$\mathbb{Z}/3$
29	$(\mathbb{Z}/2)^3$
31	$\mathbb{Z}/9$
37	$\mathbb{Z}/37$
41	$(\mathbb{Z}/11)^2$
43	$\mathbb{Z}/211$
47	$\mathbb{Z}/5 \oplus \mathbb{Z}/139$

5. WHITEHEAD TORSION

We take a slight detour to define the Whitehead torsion which was mentioned in Section 1.2.3. The idea is to define an algebraic invariant called *torsion* to a contractible chain complex (sometimes called an acyclic or exact chain complex). This torsion will correspond to the torsion defined in Section 4.2. Once that is done, we will show how to associate to any homotopy equivalence a contractible chain complex, which will allow us to the define the torsion of the homotopy equivalence.

Let (C_*, ∂) be a contractible chain complex of based Λ-modules and let $\Gamma\colon 0 \simeq \mathrm{Id}$ be a chain contraction. Recall that a chain contraction is a map $\Gamma\colon C_* \to C_*$ which satisfies $\partial\Gamma + \Gamma\partial = \mathrm{Id} - 0 = \mathrm{Id}$ (the below commutative diagram may be useful for conceptualising this).

$$\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r, "\partial"] \arrow[dl] & C_2 \arrow[r, "\partial"] \arrow[dl, "\Gamma"'] & C_1 \arrow[r, "\partial"] \arrow[dl, "\Gamma"'] & C_0 \arrow[r] \arrow[dl, "\Gamma"'] & 0 \\
\cdots \arrow[r] & C_3 \arrow[r, "\partial"'] & C_2 \arrow[r, "\partial"'] & C_1 \arrow[r, "\partial"'] & C_0 \arrow[r] & 0
\end{tikzcd}$$

We define a map

$$\partial + \Gamma := \begin{bmatrix} \partial & & & \\ \Gamma & \partial & & \\ & \Gamma & \partial & \\ & & \ddots & \ddots \end{bmatrix} : C_1 \oplus C_3 \oplus C_5 \oplus \cdots \to C_0 \oplus C_2 \oplus C_4 \oplus \dots$$

We will use the shorthand $C_{\mathrm{odd}} = C_1 \oplus C_3 \oplus C_5 \oplus \dots$ and $C_{\mathrm{even}} = C_0 \oplus C_2 \oplus C_4 \oplus \dots$ for brevity.

Exercise 5.1. Show that

$$(\partial + \Gamma)^{-1} = \begin{bmatrix} 1 & & & \\ \Gamma^2 & 1 & & \\ & \Gamma^2 & 1 & \\ & & \ddots & \ddots \end{bmatrix}^{-1} \begin{bmatrix} \Gamma & \partial & & \\ & \Gamma & \partial & \\ & & \Gamma & \ddots \\ & & & \ddots \end{bmatrix}$$

using the fact that Γ is a chain contraction.

The above means that $\partial + \Gamma \in \mathrm{GL}(\Lambda)$ so the torsion $\tau(\partial + \Gamma)$ is defined. We would like to define the torsion of C_* to be this, but we have made a choice of a chain contraction. The following lemma shows that this does not matter.

Lemma 5.2. *Let Γ and Γ' be two chain contractions of C_* and let*

$$\Delta_i = \Gamma(\Gamma' - \Gamma)\colon C_i \to C_{i+2}$$

which produces chain maps $\Delta\colon C_{\mathrm{odd}} \to C_{\mathrm{odd}}$ and $\Delta_{\mathrm{even}}\colon C_{\mathrm{even}} \to C_{\mathrm{even}}$. Then

$$\partial + \Gamma' = (1 + \Delta_{\mathrm{even}})(\partial + \Gamma)(1 + \Delta_{\mathrm{odd}})^{-1}$$

with

$$\tau(1 + \Delta_{\mathrm{even}}) = \tau(1 + \Delta_{\mathrm{odd}}) = 0.$$

Proof. For the second fact, it is not too hard to see that both $(1 + \Delta_{\mathrm{even}})$ and $(1 + \Delta_{\mathrm{odd}})$ can be reduced to the identity matrix by a sequence of row and column operations and hence their torsions are trivial. □

Definition 5.3. Let (C_*, ∂) be a contractible chain complex of based Λ-modules and Γ a chain contraction. Then we define the *torsion* of C_* as $\tau(C_*) = \tau(\partial + \Gamma)$. By Lemma 5.2 this does not depend on the choice of chain contraction Γ and hence is well-defined.

Now let $f\colon X \xrightarrow{\simeq} Y$ be a homotopy equivalence of finite CW complexes and let $\pi := \pi_1(Y)$. The map f then induces a chain equivalence of $\mathbb{Z}[\pi]$-modules $f_*\colon C_*(\widetilde{X}) \to C_*(\widetilde{Y})$ where $\widetilde{X}$ and $\widetilde{Y}$ denote the universal covers of X and Y.

Choose bases for $C_*(\widetilde{X})$ and $C_*(\widetilde{Y})$ and let $\mathscr{C}(f_*)$ be the algebraic mapping cone of f_*. Recall that given any chain map $\varepsilon\colon (A_*, \partial_A) \to (B_*, \partial_B)$ the algebraic mapping cone $\mathscr{C}(\varepsilon)$ is the chain complex

$$\mathscr{C}(\varepsilon)_k = \left(A_{k-1} \oplus B_k, \begin{bmatrix} \partial_A & \mathbf{0} \\ (-1)^k \varepsilon & \partial_B \end{bmatrix} \right).$$

It is a standard fact that the mapping cone of a chain equivalence is chain contractible.

We would like to define $\tau(f) = \tau(\mathscr{C}(f_*))$ using Definition 5.3, but this is not well-defined in $K_1(\mathbb{Z}[\pi])$ since our choice of bases for $C_*(\widetilde{X})$ and $C_*(\widetilde{Y})$ affect the result. However, it is not too difficult to see that our choices only differ by multiplication by ± 1 and elements of π, therefore we see that $\tau(f)$ is well-defined in $K_1(\mathbb{Z}[\pi])/\{\tau(\pm g) \mid g \in \pi\} = \mathrm{Wh}(\pi)$. (One might also worry about the choice of base-points for $\widetilde{X}$ and $\widetilde{Y}$, but a change of base-points will only change the torsion in $K_1(\mathbb{Z}[\pi])$ by conjugation, but $K_1(\mathbb{Z}[\pi])$ is abelian by Lemma 4.3 and hence this has no effect).

Definition 5.4. We now define the *Whitehead torsion* of a homotopy equivalence f as above to be $\tau(f) := \tau(\mathscr{C}(f_*)) \in \mathrm{Wh}(\pi)$.

From this perspective, the definition of the Whitehead group comes about very naturally.

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